Fractions: Pythagorean Tuning

Key Learning Content

This film sets out Pythagoras' fundamental insight into musical harmony: that if a stretched string vibrates to produce a note, then the notes that sound in harmony with the original note correspond to divisions of the string by whole numbers. Thus, strings half the length, a third of the length, a quarter of the length and so on would sound harmonious; strings of other lengths would not. It is the relationship between fractions that makes sound musical. Understanding the full complexity of musical theory is not necessary to view the film, although prior knowledge of fractions is necessary.

Core Outcomes

Learning Points

- Be able to calculate a given fraction of a given quantity, expressing the answer as a fraction.
- Be able to multiply and divide a given fraction by an integer, by a unit fraction, and by a general fraction.
- Be able to use ratio notation including reduction to its simplest form and its various links to fraction notation.
- Be able to divide a quantity in a given ratio, e.g. divide a string into two parts in the ratio 3:2.

Suggested Activities

- Cut lengths of string 30cm long into lengths which would produce a note harmonious with the original, and then work out the different ratios between the lengths.
- Derive the Pythagorean tuning scale using only the relationships between a string, a half-string and a third-string.

Extension Outcomes

Learning Points

- Be able to understand the relationship between musical harmony and unit fractions.
- Be able to analyse alternative musical scales in terms of their underlying mathematics.

Suggested Activities

- Test Pythagoras' contention about harmonious sounds with different musical instruments.
- Research the mathematics behind alternative musical scales and listen to the sounds they make.

Pythagoras determined that the chords that sound most melodic match precise divisions or exact fractions of the whole string.

Related Films

To use before the lesson plan:

The Egyptians and Unit Fractions	This film explores the limited use of fractions in ancient Egypt.

To use after the lesson plan:

Fractions: Slow Motion	This film shows how fractional speeds can create the special effects seen in films.
What Do Sine Waves Sound Like?	This film provides a mathematical analysis of what makes music, exploring the link between trigonometry and music through the amplitude and frequency of sound waves.
Irrational Numbers: Pythagoras	This film details how the impossibility of writing some numbers as fractions caused problems for the Greeks.
Proving Pythagoras	This film discusses the proof of the famous Pythagoras' Theorem concerning right-angled triangles.
The History of the Golden Ratio	This film explores mathematical patterns in art and music.

Guide Lesson Plan

Introduction

Using musical instruments or the web, play musical notes to the students and ask them: Which combinations of notes sound harmonious and which do not? Ask them if they have an idea why this is. Play examples of western and eastern music with different harmonics and ask them what they think are the differences between the pieces of music.

Show Film

Fractions: Pythagorean Tuning

Main Activity

Foundation

Give students string and a strong shoe box. Get them to stretch different lengths of string diagonally across the box so as to produce harmonious sounds. Get them to list all the lengths cut from a 30cm length of string that would produce notes in harmony with the 30cm string. Then ask them to calculate the ratios between all the lengths and record them in their lowest form in a square grid, with lengths across the top and down the side. Ask the students how many distinct ratios there are.

Main Activity cont ...

Next, show students the ratios between the notes in a scale constructed using Pythagorean tuning (1:1, 9:8, 81:64, 4:3, 3:2, 27:16, 243:128, relative to base note). Show how these intervals can be constructed using only the relationships between a string, a half-string and a third-string:

- Let the original string length be 1. Aim to construct notes in the interval between 1 and 2.
- Calculate the relationship between the third-string and the half-string (x 3/2) and apply this to the string of length 1 to get a string of length 3/2.
- Then repeat to get a string of length 3/2 x 3/2 = 9/4. Since this is greater than two, drop down an octave to 9/8.
- Then repeat to get a string of length 3/2 x 9/8 = 27/16, then again to get 81/32. Since this is greater than two, drop down an octave to 81/64.
- Then repeat to get a string of length 81/64 x 3/2 = 243/128.
- To get the final note, divide the string of length 1 by 3/2 to get 2/3. Since this is less than 1, double to go up an octave to 4/3. This completes the scale.

Get students to calculate the multipliers between consecutive ratios in the list 1:1, 9:8, 81:64, 4:3, 3:2, 27:16, 243:128 – what pattern do they notice?

Advanced

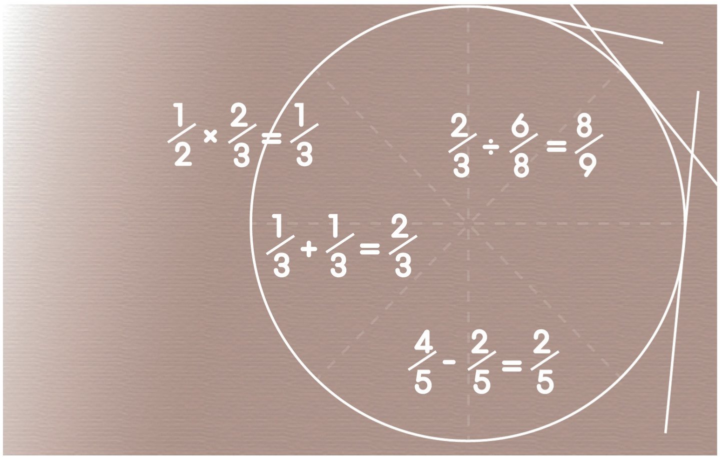
As above, but experiment with musical instruments to produce notes at different string length (or equivalent). Get students to create alternative scales and listen to what they sound like.

Extension Activity

Explain that Pythagoras built his scale based on relationships formed by dividing by only 2 and 3. Explain that there is no mathematical reason why he could not use quarter-strings and fifth-strings. What difference would it make to scales if these relationships were used? Find examples of other scales used in different cultures and explore the mathematics behind them.

Optional Extra

Build a basic musical instrument using strings stretched across a box, working to the dimensions on Pythagoras' scale.



Modern fractions are simply numbers where the top 'numerator' is divided by the bottom 'denominator'.