Fractals: The Menger Sponge

Key Learning Content

This film describes the iterative construction of a fractal shape, The Menger Sponge. No introduction is given to fractals, and the film can stand alone, but some prior knowledge of fractals would probably be useful. Using graphic images rather than numbers and algebra, the film shows how the volume of the sponge decreases to zero with each iteration, while its surface area increases to infinity. The mathematics of the Menger Sponge is complicated and probably best avoided with younger students, although the concepts introduced are accessible to all ages.

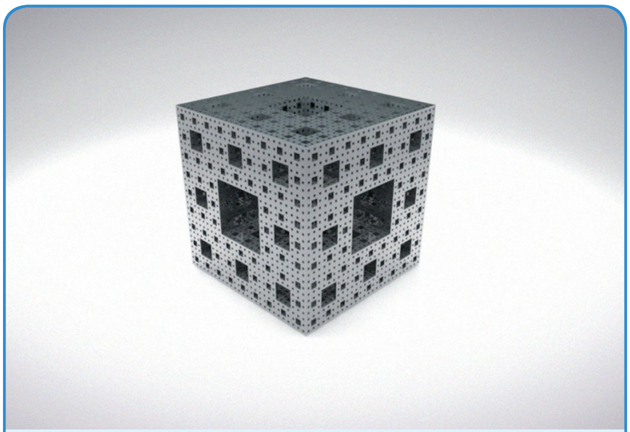
Core Outcomes

Learning Points

- Be able to understand the nature of a fractal, and its self-similarity.
- Be able to understand the terms 'face', 'edge' and 'vertex' in the context of a three-dimensional solid.

Suggested Activities

- Construct 1D and 2D analogues of the Menger Sponge, the Cantor Set and the Sierpinski Carpet.
- Establish corresponding results for these fractals in terms of number of points, length, perimeter and area.



The curious aspect of the Menger Sponge is that as the volume decreases, the surface area increases.

Extension Outcomes

Learning Points

- Be able to provide reasons, using standard geometrical statements, to support numerical values for areas and volumes in geometrical problems, and relate this to the properties of similar shapes.
- Be able to understand that areas and volumes of similar figures are in the ratio of, respectively, the square and cube of corresponding sides.
- Be able to generate terms of a sequence using position-to-term definitions of the sequence, using powers.
- Be able to interpret information presented in a non-linear graph, and relate this to the algebraic form of the functions being plotted (powers and reciprocals).

Suggested Activities

- Explore what happens to the Menger Sponge by considering the number of cubes (smaller cubes at each iteration) at each step in the process. Derive a formula for the number of cubes after each iteration (20 raised to the power n)
- Plot graphs of the surface area and volume of a single cube at each stage of the Menger Sponge's creation (more, smaller cubes at each step) and the number of cubes at each step, and relate these graphs to the graph shown in the film.

Related Films

To use before the lesson plan:

Fractals: The Koch Snowflake

This film is similar to 'The Menger Sponge' but it provides a high-level introduction to fractals and explores a 2D fractal shape.

To use after the lesson plan:

Sets: Infinity

This film introduces the history of infinity paradoxes, and provides an overview of how modern mathematicians try to treat infinity as just another number.

The Incredible Strength of Ants

This film looks at what happens to the surface area and volume of a cube as it is enlarged, using this to explain why ants can lift more than their own body weight.

Topology

This film gives examples of shapes and solids that can be logically described, but which can appear to be counter-intuitive.

Guide Lesson Plan

Introduction

Fractals are a very visual part of mathematics and perhaps the best way to introduce them is to show examples – simply typing 'fractal' into an image search engine on the web will provide dozens of stunning images. If possible, show an example of a Mandelbrot Set and the self-similarity it exhibits on repeated magnification. Then ask: Could this process go on indefinitely, with ever greater levels of detail at a smaller and smaller level?

Show Film

Fractals: The Menger Sponge

Main Activity

Foundation

The mathematics of the Menger Sponge are challenging, so approach the topic through first the 2D, then the 1D analogues of the Sponge, namely the Sierpinski Carpet (2D) and the Cantor Set (1D). Show the first stage of the Sierpinski Carpet, with a middle square cut out of a larger square, the side of the middle square being one-third the size of the larger. Get students to repeat the process and work out the area and perimeter of the Sierpinski Carpet at each step. Then do the same for a straight line, cutting out the middle third but leaving the end points of the cut (one-third and two-thirds along) as part of the remaining shape. Ask students to generalise the Menger Sponge result for the Sierpinski Carpet and Cantor Set. (With the Cantor Set, the remaining length tends to zero but the number of remaining points tends to infinity).

Main Activity cont ...

Advanced

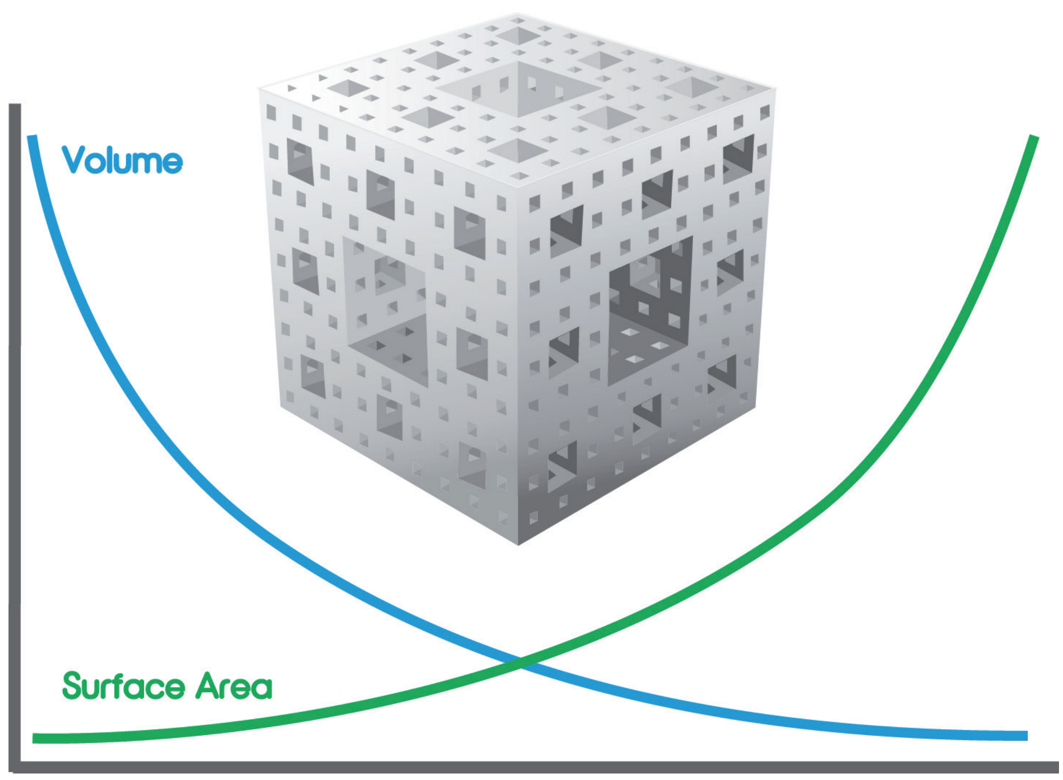
Get students to consider the Menger Sponge at each step of its creation and count the number of cubes (smaller cubes at each iteration) at each step in the process. Encourage them to look for a pattern and generalise their result (number of cubes after n iterations = 20 raised to the power n).

Extension Activity

Consolidate the students' results and ask them: What would graphs of their results look like? Show linear, quadratic, cubic and exponential graphs and their reciprocals using graphing software, and link each sequence to its corresponding graph. Then ask them to explain in general terms the shape of the graph shown in the film (e.g. for the Menger Sponge, the number of cubes increases rapidly but each individual cube's area and volume decrease rapidly; the volume decrease is greater than the increase in the number of cubes, so overall volume falls, while the increase in the number of cubes is greater than the decrease in surface area, so overall surface area increases).

Optional Extra

On the internet, find a Mandelbrot Set generator and get students to experiment with generating fractals.



The surface area of the Menger Sponge increases towards infinity, as the volume gets closer and closer to zero.