# Hyperbolic Geometry

## Key Learning Content

This film begins with an outline of Euclid's simple and intuitive approach to geometric reasoning but then asks: Is this the geometry that best describes our world? Alternative geometries are then described: first, Elliptical geometry, describing the surface of a sphere; then Hyperbolic geometry, describing the curved space of modern physics. The relationships of Hyperbolic Trigonometry are stated and contrasted with those of traditional Euclidean geometry. The complexities of Hyperbolic geometry do not need to be fully understood by the viewer, but serve to highlight those aspects of Euclidean geometry that we might otherwise take for granted.

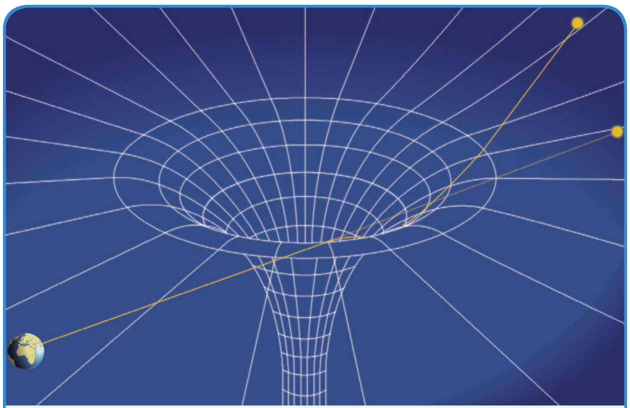
## Core Outcomes

### Learning Points

- Be able to understand what is meant by a parallel line.
- Be able to understand what is meant by a parallel line in curved space.
- Be able to understand and use Pythagoras' theorem in two dimensions.
- Be able to use sine, cosine and tangent and to understand the relationships between them.

### Suggested Activities

- Explore the idea of a parallel line on a flat piece of paper and on the surface of a globe.
- Use a Poincaré disk to explore the idea of a straight line in curved (hyperbolic) space.
- Establish the standard relationships between sin, cos and tan from first principles.



**Euclidean geometry has one parallel line, Elliptical geometry has none.**

## Extension Outcomes

### Learning Points

- Be able to understand that Euclidean geometry is just one of several geometries in mathematics.
- Be able to understand that Elliptical geometry describes the surface of a sphere.
- Be able to understand that there are no parallel straight lines in Elliptical geometry.
- Be able to understand that modern physics assumes that space is curved and uses Hyperbolic geometry and trigonometry.

### Suggested Activities

- Review Euclid's axioms and understand the significance for geometry of the 'parallel axiom'.
- Review statements that are mathematically equivalent to the parallel axiom and consider what our experience would be like if these statements were not true.
- Construct triangles with sum of angles greater than 180 degrees on the surface of a sphere.
- Construct triangles with sum of angles less than 180 degrees on a Poincaré disk.

## Related Films

To use before the lesson plan:

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| **Geometry: Euclid** | This film describes the traditional view of geometry originally taught by Euclid, which remained unchallenged for two thousand years. |
| **Measuring the Earth** | This film shows how it is possible to estimate the circumference of the earth using Euclidean trigonometry. |

To use after the lesson plan:

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|---|---|
| **Topology** | This film explores the weird and wonderful shapes that challenge our powers of perception, even in Euclidean space. |
| **Number Theory: Gauss** | This film describes the work of the mathematician who measured the curvature of space. |

## Guide Lesson Plan

### Introduction

Ask the students how many degrees there are in a triangle. Ask them if it is ever possible to draw a triangle whose angles add up to more or less than 180 degrees. Ask how they would prove their view.

### Show Film

**Hyperbolic Geometry**

### Main Activity

**Foundation**

Start by defining a parallel line: draw a straight line on the board, then add a point anywhere on the board not on the line. A parallel line is any straight line through the point that does not meet the first line, extended indefinitely. Agree that there is only one parallel line. Then do the same thing on a sphere, discussing first what is meant by a straight line on a sphere. If a straight line is the shortest distance between any two points, then all straight lines are great circles, like the equator. Agree that any two great circles on a sphere must intersect, hence there are no parallel lines. Finally, construct a triangle on the sphere with its base along the equator, and its vertex at the north pole. Ask: What is the sum of the angles in the triangle? (Answer: greater than 180 degrees).

## Main Activity cont...

**Advanced**

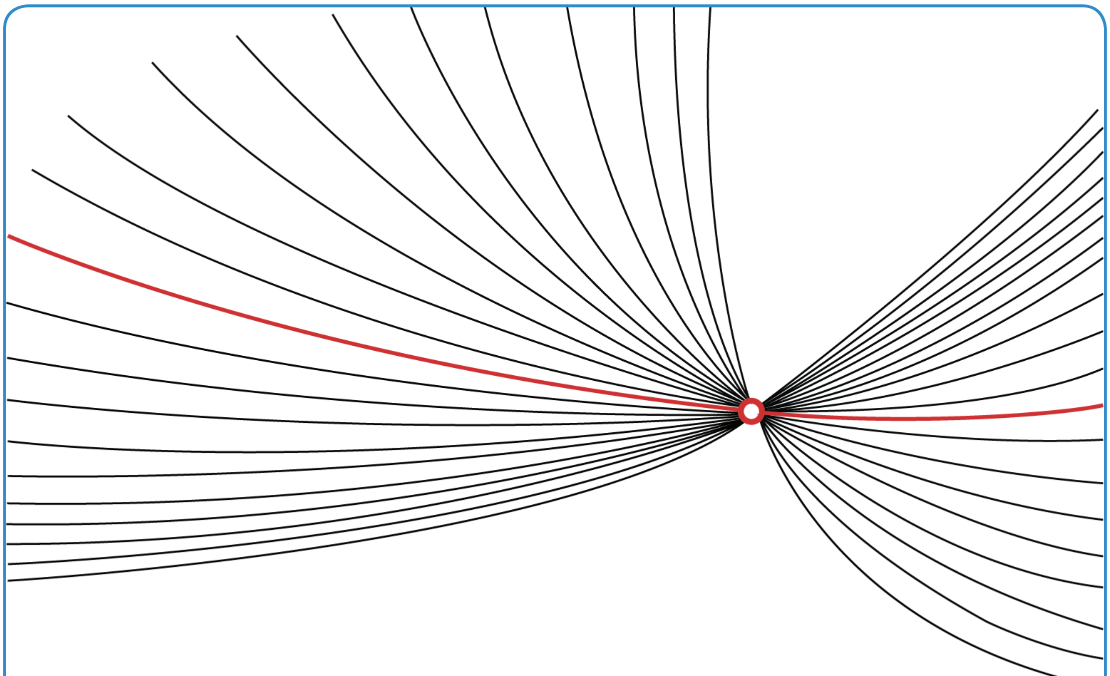
Hand out a list of Euclid's five basic axioms and discuss what each means. Ask if anyone disagrees with any. Then outline the difficulties mathematicians had in trying to prove the parallel axiom from the other four. (State the parallel axiom as: Given a line l and a point not on l, there is only one line which contains the point, and is parallel to l.) Review statements equivalent to the axiom. Try to imagine a world where these statements were not true. Then rewrite the parallel axiom with 'there is no line…' then with 'there are many lines…' and explain that these define Elliptical geometry, and Hyperbolic geometry respectively.

## Extension Activity

Give students access to an interactive model of the Poincaré disk, a Euclidean model of hyperbolic or curved space, where all space is confined within the circumference of a circle and straight lines appear curved (search the web for a Poincaré disk applet). Get students to construct multiple lines through a point parallel to a given line, and triangles with sum of angles less than 180 degrees.

## Optional Extra

Go over the definitions of sin, cos and tan and establish that tan θ = sin θ/cos θ. Draw a right-angled triangle with hypotenuse of length 1 and use Pythagoras to prove that sin2 θ + cos2 θ = 12. Give the alternative result for hyperbolic functions sinh θ and cosh θ. Explore values of sinh θ and cosh θ using a scientific calculator.



**Bolyai and Lobachyevski invented a new type of trigonometry, which could describe the relationship between lines and angles in a hyperbolic space that curves from a single point.**