Proofs: Million-Dollar Maths

NAME: ....

CLASS: ....

DATE: ....

Basic

1) In maths, what do we call assertions that are not known to be true or false?

2) What do we call the process of establishing the truth of an assertion?

3) Write down the negation of the following statements:

a) $23 \geq 34$

b) $3x + 5x = 6x$

4) Given that $6x - 8 = 2x$, prove that $x = 2$.

5) Read the following proof. If there is a fault, state the line number and why the line is wrong.

Given $x \in$ Z and $x^2 + 2x = 2x + 25$, prove $x = 5$.

$x^2 = 25$

$x = 5$

If the given statements are true, then $x = 5$.

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Core

1) Prove by contradiction that $\sqrt{2}$ is not rational.

2) Prove by induction that $n^3 + 2n$ is divisible by 3, $n \geq 1$, $n \in N$.

3) Prove that $\sin 2x = 2\sin x \cos x$.

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Advanced

1) A: n^2 is even.
   B: n is even ($n \in \mathbb{Z}$).
   Prove A $\Rightarrow$ B by contradiction.

2) Prove by contradiction that $\sqrt{7}$ is irrational.

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ANSWERS

Basic

1) Conjectures or hypotheses

2) A proof

3) a) $2^3 < 3^4$ T

b) $3x + 5x \neq 6x$ ($x \neq 0$) T

4) $4x = 8$, therefore $x = 2$; thus proved.

5) Wrong, because x = +-5.

Core

1) Assume that $\sqrt{2} \in Q$.
$\exists a, b \in Z$
$a = b\sqrt{2}$
$a^2 = 2b^2$
a^2 is even.
a is even, so a has a factor of 2.
Let a = $2k$ where $k \in Z$.
Then $4k^2 = 2b^2$ (since $a^2 = 2b^2$).
$b^2 = 2k^2$
b^2 is even.
b is even, so b has a factor of 2.
Both a and b are even.
$\Rightarrow$ a and b have a common factor of 2.
We have a contradiction; thus $\sqrt{2}$ is not rational.

2) a) The statement is true for the lowest required value of n, namely $n = 1$: $1^3 + 2 \times 1 = 3$, which is divisible by 3.
b) Assume that it is true for some value $n = k$. Therefore, $k^3 + 2k = 3m$ for some $m \in N$.
Consider $n = k + 1$.
$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$
$= k^3 + 2k + 3k^2 + 3k + 3$
$= (k^3 + 2k) + 3(k^2 + k + 1)$
$= 3m + 3(k^2 + k + 1)$
$= 3(m + k^2 + k + 1)$
$= 3m_1$ for some m1 $\in$ N
$\Rightarrow$ the statement is true for $k + 1$.
c) Since the statement is true for $n = 1$ and since (true for $n = k$) $\Rightarrow$ (true for $n = k + 1$) then, by induction, it is true $\forall$ n $\geq$ 1, $n \in$ N.

3) $\sin 2x = \sin(x + x)$
$= \sin x \cos x + \cos x \sin x$
$= 2\sin x \cos x$

ANSWERS

Advanced

1) Assume that A is true but B is false. Then n is odd and $n = 2m + 1$ where $m \in Z$.

But $(2m + 1)^2 = 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$

$= 2r + 1$

where $r \in Z$

This shows that $(2m + 1)^2 = n^2$ is odd, which contradicts statement A. Thus n must be even.

2) Assume $\sqrt{7}$ is rational.

Then $\sqrt{7} = \frac{m}{n}$ for integers m and n, which have no common factors because we can cancel any common factors first, [greatest common factor $(m, n) = 1$].

$$\sqrt{7} = \frac{m}{n}$$

$$7 = \frac{m^2}{n^2}$$

$7n^2 = m^2$

$7|m$ by the fundamental theorem of arithmetic

$7^2|m^2$

$7^2|7n^2$

$7|n^2$

$7|n$ by the fundamental theorem of arithmetic

$7|m$ and $7|n$ i.e. m and n have a common factor.

This contradicts $(m, n) = 1$.

This in turn contradicts our original assumption that $\sqrt{7}$ is rational, and hence $\sqrt{7}$ must be irrational.